Optimization

Last Time: Gradient = vector in direction of maximized line for f
for $f: \mathbb{R}^n \to \mathbb{R}$ $\quad \vec{p} \in \mathbb{R}^n$ is a critical point of f when
$\nabla f(\vec{p}) = \vec{0}$ or DNE

Fermot's Extrema Theorem: If f has a local extrema at $\vec{p}$, then $\vec{p}$ is a critical point of f

The Extreme Value Theorem: If f is defined on a closed and bounded subset of $\mathbb{R}^n$, then f obtains extreme values on k. (In other words, on k there is an absolute max and absolute min for f.)

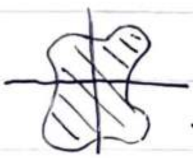
What is "closed" and "bounded"?

In $\mathbb{R}^1$, this just means "union of finitely many closed interval

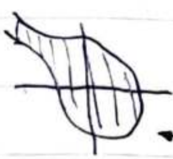
Bounded

Not Bounded

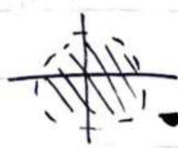
In $\mathbb{R}^2$



← is closed and bounded



← Not Bounded



← Not closed

Closed also means:

"the boundary between the set at the rest is part of the set itself"

In Calc III, the closed interval method of Calc I becomes:

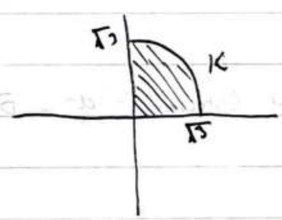
<u>Compact Set Method</u>:

- Suppose K is closed and bounded and f goes from $K \to \mathbb{R}$. To optimize f on K,
  1. Compute the critical points of f on K
  2. Find extreme values among critical points
  3. Optomize along the boundary

The extreme values are the largest and smallest of these computed

<u>Example</u>: Compute the global extreme values of $f(x,y) = xy^2$ on $K = \{(x,y) : 0 \le x,\ 0 \le y,\ x^2+y^2 \le 3\}$

picture:



Sol: Find critical points

$\nabla f = \langle y^2, 2xy \rangle$

$\nabla f = 0$ iff $\langle y^2, 2xy \rangle = \vec{0}$

iff $\begin{cases} y^2 = 0 \\ 2xy = 0 \end{cases}$ iff $\begin{cases} y = 0 \\ x = 0 \text{ or } y = 0 \end{cases}$ iff $y = 0$

Now we need to optimize the boundary and the critical points.

In this example, all the critical points are all located on the boundary

Parametreize the Boundary (to optimize it)

three curves make the boundary:

$$\left.\begin{aligned} b_1(t) &= (t, 0) \quad \text{for} \quad 0 \le t \le \sqrt{3} \\ b_2(t) &= (0, t) \quad \text{for} \quad 0 \le t \le \sqrt{3} \\ b_3(t) &= (\sqrt{3}\cos t, \sqrt{3}\sin t) \quad \text{for} \quad 0 \le t \le \tfrac{\pi}{2} \end{aligned}\right\} \quad \text{Look at } f(b_i(t))$$

On b_1 : $f(b_1(t)) = f(t, 0) = t \cdot 0^2 = 0$

On b_2 : $f(b_2(t)) = f(0, t) = 0 \cdot t^2 = 0$

On b_3 : $\underbrace{f(b_3(t))}_{g(t)} = f(\sqrt{3}\cos t, \sqrt{3}\sin t) = \sqrt{3}\cos t \cdot (\sqrt{3}\sin t)^2 = 3\sqrt{3}\cos(t)\sin^2(t)$

$g'(t) = 3\sqrt{3}(-\sin t)\sin^2 t + \cos(t)(2\sin(t)\cos(t)) = 3\sqrt{3}\sin(t)(2\cos^2 t - \sin^2 t)$

$\therefore g'(t) = 0$ iff $\sin(t) = 0$ or $2\cos^2 t - \sin^2 t = 0$

iff $t = k\pi$ for some integer k

or $2\cos^2(t) = \sin^2(t)$

iff $t = k\pi$ for some integer k

or $2 = \tan^2(t)$

iff $t = k\pi$ for some integer k

or $\tan(t) = \pm\sqrt{2}$

iff $t = k\pi$ for some integer k

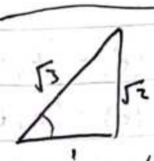
or $t = \arctan(-\sqrt{2})$

or $t = \arctan(\sqrt{2})$

On $0 \le t \le \frac{\pi}{2}$, this means $t = 0$ or $t = \arctan(\sqrt{2})$

$\therefore \quad g(0) = 0 \;,\; g(\frac{\pi}{2}) = 0 \;,$ and $g(\arctan(\sqrt{2}))$: ?

$$g(\arctan(\sqrt{2})) = 3\sqrt{3}\, \cos(\arctan(\sqrt{2}))\, \sin^2(\arctan(\sqrt{2})) > 0$$
$$= 3\sqrt{3}\, \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{2}}\right)^2 \cdot 2$$



$\theta = \arctan(\sqrt{2})$

$\frac{\sqrt{2}}{1} = \tan\theta$

$\cos\theta = \frac{1}{\sqrt{3}}$

$\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$

$\therefore$ The extreme values of f are :

0 is global min on K

2 is global max

<u>Q</u> : For local optimization, are there good first and second derivative tests for calc III ?

<u>A</u>: Yes, but they are more complicated ...

<u>First Derivative Test</u>:

Suppose f is diff at p

① If for all suff small $\varepsilon > 0$ and all unit vectors $\vec{u} \in \mathbb{R}^n$,

$D_{\vec{u}} f(\vec{p} + \varepsilon\vec{u}) > 0$, then f has a local min at $\vec{p}$

② If for all suff small $\varepsilon > 0$ and all unit vectors $\vec{u} \in \mathbb{R}^n$,

$D_{\vec{u}} f(\vec{p} + \varepsilon\vec{u}) < 0$, then f has a local max at $\vec{p}$

We'll get a second derivate test, but this will only apply to functions of 2 variables ...

For a function $f(x,y)$, the function $D = \det\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx} \cdot f_{yy} - f_{xy}\, f_{yx}$
$= f_{xx} \cdot f_{yy} - f_{xy}^2$

is important

it's an analogue of second derivative

Second Derivative Test: Suppose f is diff at $\vec{p}$ and $\vec{p}$ is a C.P.

① If $f_{xx}(\vec{p}) > 0$ and $D(\vec{p}) = f_{xx}(\vec{p}) f_{yy}(\vec{p}) - (f_{xy}(\vec{p}))^2 > 0$
then f has a local min at $\vec{p}$

② If $f_{xx}(\vec{p}) < 0$ and $D(\vec{p}) = f_{xx}(\vec{p}) f_{yy}(\vec{p}) - (f_{xy}(\vec{p}))^2 > 0$
then f has a local max at $\vec{p}$

③ If $D(\vec{p}) = f_{xx}(\vec{p}) f_{yy}(\vec{p}) - (f_{xy}(\vec{p}))^2 < 0$
then f has a saddle point at $\vec{p}$
f crosses the tangent plane at $\vec{p}$